Shiryayev Sequential Probability Ratio Test for Redundancy Management

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An essential aspect in the design of fault-tolerant digital flight control systems is the design of failure detection and redundancy management systems. A decision rule, the Shiryayev sequential probability ratio test (SPRT), is derived from a dynamic programming approach and is used to detect failures between similar instruments, as well as between dissimilar instruments through analytic redundancy. Unlike the Wald SPRT, which tests for the presence of failure or no failure in all of the data sequence, the Shiryayev SPRT detects the occurrence of a fault in the data sequence in minimum time if certain conditions are met. The performance of the Shiryayev SPRT in detecting a failure between two rate gyros as compared to standard fixed interval schemes is presented, as well as the performance of a single accelerometer failure using translational kinematic equations to form a parity relation for analytic redundancy.

I. Introduction

THE design of reliable, fault-tolerant, digital flight control and fire control systems requires that sensor failures be detected and identified within acceptable time limits, such that system feedback are not excessively corrupted. The performance of the redundancy management scheme will be especially important for statically unstable aircraft, control configured aircraft, and some V/STOL aircraft, where the function of the autopilot is essential to the safe operation of the aircraft. The principal tradeoff to be made in designing a redundancy management scheme is that of hardware redundancy vs the complexity and robustness problems of the software for analytic redundancy (i.e., combining dissimilar instruments through analytic kinematic and dynamic relationships to obtain redundancy). Generally, two or more instruments of a particular type are available to meet control system reliability constraints, although the cost of some devices may be so great that no hardware redundancy is practical. Note that even if two instruments of the same type are available, the isolation of a failure is possible only with the use of analytic redundancy.

Our objective is to describe a decision rule which is based upon the results of Shiryayev[1] and suggested to us by Deyst.[2] Other schemes for fault detection and isolation are given in Ref. 3. The Shiryayev sequential probability ratio test (SSPRT) is a relatively simple to implement decision rule that is sensitive enough to produce reasonable failure detection performance. The SSPRT can be employed to detect a failure in two similar instruments, as well as in dissimilar instruments via analytic redundancy. In particular, the SSPRT tests for the occurrence of a disruption (i.e., failure) in a sequence of conditionally independent measurements. The detection of a failure identical to the assumed design failure occurs in minimum time[1] under certain conditions. This is in contrast to the Wald SPRT (WSPRT),[4] which simply chooses in minimum time[1] between the hypotheses that there is a failure or that there is no failure in *all* of the measurements. Since the time of failure is generally unknown, the Wald test requires a "trigger" to initiate the WSPRT.[5] The trigger is usually a direct comparison between two similar instruments that indicates the possibility of a failure. By the very nature of the SSPRT, no triggering mechanism is necessary and the test remains in continuous operation. The SSPRT will be shown to reduce to the WSPRT when the probability of a change in the failure state is zero. Furthermore, it is shown that the SSPRT responds quicker than the fixed-interval trigger used in Ref. 5 and 6.

The paper is organized as follows. In Sec. II, the essential assumptions and properties of the SSPRT,[1] derived from a dynamic programming approach, are presented. In Sec. III, explicit densities are assumed which form the basis of the numerical work of Secs. IV and V. In particular, Sec. IV formulates the parity relation that is used for residual formation in the translational kinematic analytic redundancy test. In Sec. V, numerical results on the performance of the SSPRT are given for similar instruments and compared to a fixed-interval scheme, and for a scheme using analytic redundancy without a redundancy trigger, i.e., only one instrument of a kind is necessary.

II. Derivation of the Shiryayev SPRT

The objective of this section is to determine the optimal policy for announcing that a disruption occurs in a sequence of conditionally independent measurements by using a dynamic programming analysis. (See Ref. 7 for a similar derivation of the Wald SPRT.) The optimal policy will then be rewritten in terms of the likelihood ratio, which is the ratio of the probability that a failure has occurred in the observed data sequence to the probability that a failure has not occurred. The likelihood ratio can be propagated from one time to the next by a recursion relation, which permits a straightforward sequential testing of the measurement sequence. A failure or disruption in the data sequence will be defined as a change in the probability density function of the measurements.

A. Propagation Equation for the Conditional Density of a Disruption Given the Measurement Sequence

Define the sequence of measurements up to time t_N to belong to the countable set $Z_N = \{z_1, z_2, \ldots, z_N\}$. Note that for the example discussed later the measurements in the decision test are the residuals that remain when similar instrument outputs and/or analytic relationships between sensor outputs

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are differenced. Let the failure state be defined as x^0 if no disruption has occurred and x^1 if a disruption has occurred. Hence, if the true state is x^0 for $k=1,...,i-1$, the measurements z_k, $k=1,...,i-1$, are independent and distributed identically with probability density function $f_0(z_k)$, and if the true state is x^1 for $k=i,...,N$, then the measurements z_k, $k=i,...,N$, are independent and identically distributed with probability density function $f_1(z_k)$.

The dynamic programming algorithm requires a probabilistic state which can be propagated by a recursion relation from stage to stage. The probabilistic state is F_k, defined as the probability that a change in the distribution of the measurements occurs at or before t_k, given the data sequence $Z_k \triangleq \{z_1,...,z_k\}$. Define the unknown time of the disruption to be θ, such that

$$F_k \triangleq P(\theta \le t_k / Z_k) \tag{1}$$

Also, define p to be the probability of a change in the distribution occurring during an interval $T \triangleq (t_{k+1}-t_k)$, and π to be the a priori probability that f_1 is the true probability density function [i.e., $\pi \triangleq P(\theta \le t_0)$]. Notice that p is assumed to be a constant and, therefore, the time of failure is geometically distributed. By the induction argument given in Appendix A, the recursion relation for propagating F_k is

$$F_{k+1} = \frac{[F_k+p(1-F_k)]f_1(z_{k+1})}{[F_k+p(1-F_k)]f_1(z_{k+1})+[(1-p)(1-F_k)]f_0(z_{k+1})}$$

$$F_0 = \pi \tag{2}$$

and has a central role in the following dynamic programming algorithm.

B. Dynamic Programming Formulation of the Shiryayev SPRT

At each time stage the Shiryayev test allows for one of two possible actions to be taken:

1) Terminate taking measurements and announce that a disruption has occurred. There is zero cost incurred if this decision is correct, but Q is the cost incurred if the choice is incorrect.

2) Continue taking measurements with an additional cost $C>0$. The optimal decision rule is determined by minimizing the expected cost.

Our analysis first assumes that a maximum of N measurements are taken and that at stage N a failure is announced so that the test is terminated. This procedure allows easy derivation of some of the properties of this test which must hold in the steady-state case (N goes to infinity) where this test is generally applied.

The optimal expected cost for the last stage is

$$J_N^*(F_N) = (1-F_N)Q \tag{3}$$

which is the cost of a false alarm due to termination. By the dynamic programming algorithm of Ref. 7 the expected cost for the next to last stage

$$J_{N-1}^*(F_{N-1}) = \min[(1-F_{N-1})Q, \quad C+\underset{z_N}{E}(J_N(F_N)/Z_{N-1})] \tag{4}$$

where $(1-F_{N-1})Q$ is the cost of a false alarm and the second term is composed of the cost of taking one more measurement and the expected cost associated with a false alarm. Note that $E(\cdot/Z_{N-1})$ denotes expectation over z_N conditioned on Z_{N-1}. In general,[7] the expected cost can be written as

$$J_k^*(F_k) = \min[(1-F_k)Q, \quad C+A_k(F_k)] \tag{5}$$

with

$$A_k(F_k) \triangleq \underset{z_{k+1}}{E}(J_{k+1}^*(F_{k+1})/Z_k) \tag{6}$$

where the expectation is taken with respect to the conditional probability density function

$$f(z_{k+1}/Z_k) = (1-p)(1-F_k)f_0(z_{k+1}) + [F_k+p(1-F_k)]f_1(z_{k+1}) \tag{7}$$

and Eq. (2) is used in Eq. (6).

The optimal decision policy is developed using the fact that the functions A_k are concave with respect to the probability F (subscript k is droped for generality) and additionally satisfy the conditions that

$$\text{I.} \quad A_k(1) = 0 \text{ for } k=1,2,...,N-1 \tag{8}$$

$$\text{II.} \quad A_{k-1}(F) \ge A_k(F) \text{ for } k=2,3,..., N-1 \text{ and all } F \in [0,1] \tag{9}$$

These statements are proven in Appendix B.

Referring to Fig. 1, the *optimal* policy that minimizes the expected cost at each stage can now be written as

1) Announce a disruption if $F_k \ge F_{T_k}$

2) Take another measurement if $0 \le F_k \le F_{T_k}$ where F_{T_k} is determined at each stage by

$$(1-F_{T_k})Q = C + A_k(F_{T_k}) \tag{10}$$

and $C+A_{N-1}(0)<Q$. Obtaining the thresholds F_{T_k} by Eq. (10) is numerically complex.

This test is generally used by assuming an infinite number of stages, i.e., $N-k \to \infty$. Since the A_k's are monotonically decreasing and bounded because of the concavity property, then they have a limit and the thresholds F_{T_k} have a limit F_T as

$$F_{T_k} \le F_{T_{k-1}} \le ... \le F_T \le 1 - C/Q \tag{11}$$

The dynamic programming algorithm for infinite time reduces to

$$J^*(F) = \min[(1-F)Q, C+A(F)] \tag{12}$$

where

$$A(F) = \underset{z}{E}[J^*\{[F+p(1-F)]f_1(z)/[F+p(1-F)]f_1(z) + [(1-p)(1-F)]f_0(z)\}] \tag{13}$$

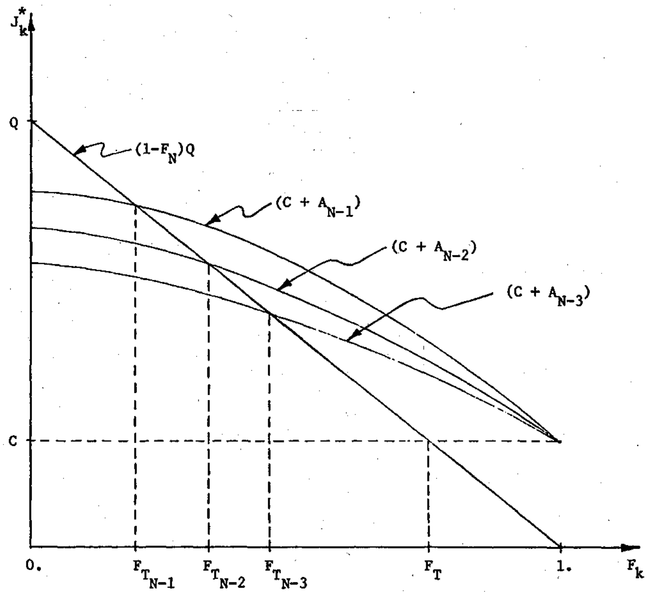

Fig. 1 The Shiryayev SPRT optimal return function.

and the threshold is determined as

$$(1-F_T)Q=C+A(F_T) \tag{14}$$

Determination of $A(\cdot)$ is quite difficult. Rather, F_T is chosen so as to imply some (Q,C) by specifying a false alaarm rate $1-F_T$.

These values have not been determined. Nevertheless, it is in the context of (Q,C) that the SSPRT gives the minimum stopping time out of the set of stopping times τ, since from Ref. 1 Eq. (12) can be stated as the minimum risk as

$$J^* = \inf_{\tau} E[(1-F_\tau)Q + C\max\{\tau-\theta,0\}] \tag{15}$$

where $E[1-F_\tau]$ is the expected false alarm probability and $E[\max\{\tau-\theta,0\}]$ is the average delay of detecting the occurrence of the disruption correctly. Our approach is to assume that the probability of false alarm $(1-F_\tau)$ is fixed at $(1-F_T)$. Then Q/C is a Lagrange multiplier and the optimization problem is to minimize the mean time of delay for detecting a failure subject to a given false alarm probability.

C. Likelihood Ratio Form of the Shiryayev SPRT

The final form of the Shiryayev decision rule and the one used in all of our simulations is obtained by defining the likelihood ratio

$$L_{k+1} \triangleq \frac{F_{k+1}}{1-F_{k+1}} \tag{16}$$

and using Eq. (2) to obtain

$$L_{k+1} = \left[\frac{f_1(z_{k+1})}{f_0(z_{k+1})}\right]\left(\frac{L_k+p}{1-p}\right) \tag{17}$$

where

$$L_0 \triangleq \frac{\pi}{1-\pi} \tag{18}$$

The Shiryayev SPRT becomes

1) Announce a disruption if $L_k \geq L_T$.

2) Take another measurement if $0 \leq L_k \leq L_T$ where the threshold is given as

$$L_T \triangleq \frac{F_T}{1-F_T} \tag{19}$$

Note that the sequential rule of Eq. (17) reduces to that of the Wald test for $p=0$ where a recursion formula results for $\lambda_k \triangleq \ln L_k$ using Eq. (17). The Wald test involves two thresholds where it is assumed that all of the data relates to one of the two hypotheses since $p=0$. Chien and Adams[9,10] noted that the Wald recursion formula can be used as a suboptimal decision rule for detecting a disruption if the tendency for λ_k to move toward the no-disruption boundary is nulled by a controller when $\lambda_k<0$. Therefore, the test will only terminate when the disruption hypothesis boundary is reached. The threshold chosen by Chien is related to the threshold given by Eq. (19). Under certain limiting conditions[9] the continuous form of Chien's suboptimal decision rule performs almost as well as the continuous form of the above Shiryayev SPRT. However, because of the simplicity of the above test it is not clear why suboptimal rules are required, especially if non-Gaussian measurements are used (see Sec. III. B).

III. Measurement Probability Densities for Implementation of the Shiryayev Test

The Shiryayev test has been applied to the detection of a bias failure where the problem is formulated as a test between two hypotheses. H_0 is the hypothesis that no failure has occurred and H_1 is the hypothesis that a failure has occurred where the failure is modeled as a known constant bias of magnitude b. Additive white Gaussian noise η with mean zero and variance σ^2 is assumed to exist under both hypotheses. Hence the detection process becomes that of testing between

$$H_0: \quad z_k = \eta_k \tag{20}$$

$$H_1: \quad z_k = \pm b + \eta_k \tag{21}$$

where η_k has a Gaussian probability density function given by

$$f(\eta_k) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{\eta_k^2}{2\sigma^2}\right] \tag{22}$$

Since the observation z_k is determined from the difference between sensor outputs and/or analytic relationships, the detection routine must take into account the possibility that the bias can be positive or negative. Two techniques for determining the ratio of the probability densities, as required by the Shiryayev SPRT, are given subsequently.

A. Two Simultaneous Hypothesis Tests

One possible implementation of the Shiryayev test simply requires that two simultaneous tests be executed, where one test checks for the possibility of a positive bias $+b$ and the second test checks for the possibility of a negative bias $-b$.

The hypotheses for testing for a positive bias are

$$H_0: \quad z_k = \eta_k \tag{23}$$

$$H_1: \quad z_k = +b + \eta_k \tag{24}$$

where the probability density functions under H_0 and H_1 are given, respectively, by

$$f_0(z_k) = f(z_k) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{z_k^2}{2\sigma^2}\right] \tag{25}$$

$$f_1(z_k) = f(z_k - b) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(z_k-b)^2}{2\sigma^2}\right] \tag{26}$$

The ratio of the probability density functions reduces to

$$\frac{f_1(z_k)}{f_0(z_k)} = \exp\left[\frac{2bz_k - b^2}{2\sigma^2}\right] \triangleq B_{k+1}^{+} \tag{27}$$

The hypothesis H_1 for testing a negative bias is obtained by replacing b in Eq. (24) with $(-b)$. B_{k+1}^{-} is obtained by replacing the minus sign in Eq. (27) with a plus sign.

The stochastic behavior of the Shiryayev test using these ratios of density functions is not easily analyzed. The test is a Markov sequence of the form

$$L_{k+1}^{\pm} = B_{k+1}^{\pm}(L_k^{\pm}+p)/(1-p) \tag{28}$$

where $B_{k+1}^{\pm}$ has a lognormal probability distribution function. This distribution function is discussed in Chap. 14 of Ref. 8. Essentially, the exponential in $B_{k+1}^{\pm}$ gives greater emphasis to positive powers and much less emphasis to negative powers. Hence, as the exponent of the exponential changes sign to a positive value due to the introduction of a sufficiently large bias and/or noise observations, the likelihood function will increase in magnitude very rapidly and may cross the threshold value, L_T.

B. Single Hypothesis Tests

The previous implementation has the obvious disadvantage of requiring the computational burden of executing two simultaneous hypothesis tests. An alternative to this approach is that of making the following single hypothesis test

$$H_0: \quad |z_k| = |\eta_k| \tag{29}$$

$$H_1: \quad |z_k| = |\pm b + \eta_k| \tag{30}$$

where the $|(\cdot)|$ denote absolute values and the noise term has the probability density given by Eq. (22). The probability density functions under H_0 and H_1 are no longer Gaussian as in Sec. III.A, but can be written as

$$f_0(z_k) = f(z_k) + f(-z_k) = \frac{2}{\sqrt{2\pi}\sigma}\exp\left[-\frac{z_k^2}{2\sigma^2}\right] \tag{31}$$

$$f_1(z_k) = f(z_k \pm b) + f(-z_k \pm b) = \frac{1}{\sqrt{2\pi}\sigma}\left[\exp\left(-\frac{(z_k+b)^2}{2\sigma^2}\right) + \exp\left(-\frac{(z_k-b)^2}{2\sigma^2}\right)\right] \tag{32}$$

The ratio of the probability density functions becomes

$$\frac{f_1(z_k)}{f_0(z_k)} = \left[\exp\left(-\frac{b^2}{2\sigma^2}\right)\right]\left(\frac{1}{2}\right)\left[\exp\left(\frac{z_k b}{\sigma^2}\right) + \exp\left(-\frac{z_k b}{\sigma^2}\right)\right] = \exp\left(-\frac{b^2}{2\sigma^2}\right)\cosh\left(\frac{z_k b}{\sigma^2}\right) \tag{33}$$

The final form of the density ratio indicates that a computational advantage lies with this implementation of the Shiryayev test. Sine the first factor is just a constant, only the hyperbolic cosine term need be calculated (in a truncated series form) in the onboard computer. The previous test implementation would require that two exponentials be calculated. However, taking the absolute value reduces the effective information, and the time to detect a disruption for a given false alarm probability is increased (see Sec. V for numerical verification). A similar test for the WSPRT is suggested in Ref. 4 where it is also required that the cosh function be evaluated.

IV. Residual Formation

Recall from Sec. II that the measurements utilized by the ratio of the density functions in the Shiryayev SPRT are actually the residuals that remain after sensor outputs and/or analytic relationships are differenced. Detection time data will be presented in the next section for similar instrument failure detection and for dissimilar instrument failure detection by analytic redundancy. Residuals are generated at each time stage for similar instrument failure detection by simply differencing the outputs from two independent sensors of the same type. The residual variance for the similar instrument case is twice the variance of the instrument output since additive white Gaussian instrument noise is assumed. Residual formation via analytic relationships for dissimilar sensor failure detection is complicated by the fact that different instruments may require different analytic relationships and that more than one form of analytic redundancy may exist. Translational kinematics analytic redundancy can be employed to detect failures in pitch rate gyros and in normal accelerometers, and is used in generating the residuals for the dissimilar sensor failure detection data of the next section. See Refs. 5 and 6 for a more general discussion.

Translational kinematics (vertical component here) analytic redundancy forms a residual by differencing the measured normal velocity and the estimated normal velocity of an aircraft. Normal accelerometer and pitch rate gyro outputs are used to form the estimated vertical velocity so failures in these instruments can be detected. Outputs from the attitude gyros, and the angle of attack and Mach meters are employed in both the measured and estimated velocity equations and are assumed unfailed. Other tests are developed which monitor faults in these sensors.[5,6] Unfortunately, the noise variance of the residual is large enough with translational kinematics to make it difficult to detect a small bias failure by using residuals formed at each stage. This problem is overcome in the case of the WSPRT by using the trigger to indicate the possibility of a failure, and then propagating the a priori estimate from one time stage to the next in forming the velocity estimate at the next stage.[5,6] The effect of this is to cause any bias to grow linearly with time. This growth expedites the Wald detection process, despite the fact that a time-varying failure threshold also must be employed, by increasing the signal-to-noise ratio. This process is not useful for the Shiryayev test since this test is in continuous operation. A bias magnitude in the velocity estimate equation can, however, be increased effectively without significantly increasing the variance of the residual by integrating the progation term over j stages, in between residual formation, such that the velocity estimate equation is

$$\hat{V}^z(t_{k+j}) = V_m^z(t_k) + \sum_{i=k+1}^{k+j}\left[\left((\bar{A}_m^z(t_i) + \bar{V}_m^x(t_i)\bar{q}_m(t_i) + x_{\text{acc}}\bar{\dot{q}}_m(t_i)\right)T\right] \tag{34}$$

where the measured normal velocity is given by

$$V_m^z(t_k) = V_s M_m(t_k)\sin[\alpha_m(t_k)] \tag{35}$$

Note that the measured velocity is propagated in the velocity estimate equation to be consistent with the SSPRT. The terms in the above equations are defined as follows: V is the velocity, V_s the speed of sound, A acceleration, q pitch rate, $\dot{q}$ pitch rate rate, x_{acc} the distance from the aircraft center of gravity to the normal accelerometers along the body fixed x axis, $T = 0.0625$ s is the sample interval, M the Mach number, α the angle of attack, t_i the time epoch, superscripts indicate a particular body axis component, the subscript m indicates that the quantity is a measured value, bars indicate quantities averaged over the current and previous time stages, and a caret denotes an estimate. Hence, the residual is formed every j stages, by differencing $\hat{V}^z$ and V_m^z.

The approximate residual variance analysis that leads to the improved signal-to-noise property of the Shiryayev SPRT produces some interesting conclusions. The work is approximate in that all instruments are assumed to have Gaussian outputs, that the product of two Gaussian random variables is assumed Gaussian with a variance equal to the second moment, and that $\sin\alpha \cong \alpha$. The first conclusion is that the measured vertical velocity variance in the estimated velocity equation is orders of magnitude larger than the variance of the propagation term. Hence, any real reduction in the residual variance must come as a result of reducing the measured velocity variance. The second conclusion is that the variance of the angle-of-attack meter affects the residual variance much more than the variances of any of the other instruments. Also, the Mach number strongly influences the variance of the failure residual. Instrument variances assumed will be given in the next section.

V. Results

The three design parameters π, F_T, and p must be chosen before the SSPRT can be implemented. The quantity π is the probability that a failure has occurred prior to the beginning of the test. The value chosen for π is 1×10^{-20} so that the initial likelihood ratio is small enough so that failures can be injected at or near the beginning of the test without affecting the detection times. This number is consistent with the likelihood ratio after 30 min of testing when no failure is present. F_T is the probability that a failure has occurred, given the observation sequence, that is required to be met or

Table 1 Similar instrument failure detection using the Shiryayev SPRT

Instrument—Pitch rate gyro, $\sigma_q = 0.04$ deg/s, $\sigma_{residual} = 0.5672$ deg/s
Design bias $\triangleq 0.3438$ deg/s, $\pi = 1\times10^{-20}$
False alarm probability $\triangleq 1\times10^{-4} \rightarrow F_T = 0.9999$
Detection times are 50-run averages.
Maximum time of each run is 30 min.
A number in parentheses is the number of runs in which a failure is detected if less than 50.

p	Actual bias (injected at $t=1$ s)	Detection time, s: Cosh test	Exp test
	0.13	108.2388	45.2688
	0.15	10.8013	7.7713
1×10^{-5}	0.19	1.6375	1.5713
	0.25	1.1563	1.1475
	0.3438	1.0438	1.0425
	0.13	(23)	(44)
	0.15	78.3275	28.8163
1×10^{-10}	0.19	2.3125	1.9713
	0.25	1.2325	1.2075
	0.3438	1.0900	1.0788
	0.13	(1)	(7)
	0.15	436.8038	117.7038
1×10^{-15}	0.19	2.8425	2.3338
	0.25	1.3500	1.3175
	0.3438	1.1288	1.1213
	0.13	(0)	(1)
	0.15	(22)	(47)
1×10^{-20}	0.19	3.4663	2.6750
	0.25	1.4500	1.3913
	0.3438	1.1688	1.1563
	0.13	(0)	(0)
	0.15	(4)	(29)
1×10^{-25}	0.19	4.2813	3.1888
	0.25	1.5075	1.4700
	0.3438	1.2063	1.2038
	0.13	(0)	(0)
	0.15	(1)	(8)
1×10^{-30}	0.19	4.6725	3.5800
	0.25	1.6375	1.6000
	0.3438	1.2600	1.2350

exceeded by the Shiryayev test before a failure is announced. Therefore, the false alarm probability is $(1-F_T)$ and is chosen to be 1×10^{-4}, consistent with Refs. 5 and 6. Recall that p is the probability of failure occurring during any one time interval T. An approximate value for p can be determined. An inertial navigation gyro failure rate of one failure in each 1.75×10^5 h is given in Ref. 9. Hence $P[k$ failures in n samples$] = \binom{n}{k}p^k(1-p)^{n-k}$, where $k=1$, $n=57{,}600$ samples/h and $\binom{n}{k}$ = number of combinations of n things taken k at a time. The above relation reduces to the probability of one failure in 1 h as

$$5.714\times10^{-6} = (57{,}600)p(1-p)^{57{,}599} \tag{36}$$

and p can be calculated to be approximately 1×10^{-10}.

Instrument models also play an important role in the implementation and performance of the Shiryayev test. The design bias level b may be a function of the possibility of unfailed sensor biases, although the robustness of the flight control system and the severity of a typical failure are more important considerations. The other instrument model parameter that strongly influences the decision process is the output variance. The conservative variance values given in Refs. 5 and 6 are also used here, except that the angle-of-attack variance has been reduced to enhance the detection of small biases when using translational kinematics analytic redundancy. The various standard deviations used are:

$\sigma_A = 0.984$ ft/s^2, $\sigma_q = 0.04$ deg/s, $\sigma_M = 0.01$ Mach, $\sigma_\alpha = 0.191$ deg.

Table 2 Small failure detection using analytic redundancy and the Shiryayev SPRT

Analytic redundancy type—Translational kinematics, instrument tested—Normal accelerometer, design bias level $\triangleq 6.4$ ft/s, $\sigma_A = 0.984$ ft/s^2, $\sigma_q = 0.04$ deg/s, $\sigma_{Mach} = 0.01$, $\sigma_\alpha = 0.191$ deg, $\pi = 1\times10^{-20}$, $p = 1\times10^{-10}$, $F_T = 0.999$, $\sigma_{residual} = 4.4$ ft/s, $t_{max} = 30$ min, failure injected at $t=0$ s, $T = 0.0625$ s
Detection times are 10 run averages for actual biases of 3 or 4; 100 run averages for actual biases of 6.4 or 12.8. A number in parentheses is the number of runs in which a failure is detected.

Residual interval	Detection time, s — Actual bias: 3.0	4.0	6.4	12.8
$30T$	(0)	172.89	20.57	6.90
$35T$	(0)	113.49	17.04	6.23
$40T$	(0)	85.15	14.90	5.03
$45T$	(0)	66.85	13.28	5.63
$50T$	(0)	55.94	11.88	5.28
$55T$	(0)	48.61	10.76	3.82
$60T$	(0)	43.35	10.46	3.75
$65T$	(0)	39.33	9.83	4.06
$70T$	(0)	35.13	9.14	4.38
$75T$	(0)	32.63	9.28	4.69
$80T$	(0)	31.10	9.35	5.00

A. Similar Instrument Failure Detection

Table 1 gives some ensemble-average (50 runs) detection time results for the Shiryayev SPRT operating on the difference between two pitch rate gyro outputs. The pitch rate gyro model is identical with that used in Refs. 5 and 6, where it is assumed that the gyro can have an unfailed bias of 0.1719 deg/s. Since the decision process is based on the difference between two gyro outputs, a bias that is less than 0.1719 deg/s implies that no failure is present, while a bias greater than 0.3438 deg/s means that a failure has occurred. A bias in the residuals that lies between these levels is inconclusive. The bias level designed into the SSPRT for the data of Table 1 is 0.3438 deg/s. The nature of the Shiryayev SPRT is to tend to not detect biases below a level of one-half the design level, while providing delayed detection of failure biases with levels between the design bias and one-half of the design bias. The data of Table 1 show the variation in detection times for both forms of the probability density function ratio given in Sec. III as a function of p and for five different bias levels. The failure is injected at $t=1$ for all of this data, so the performance of the SSPRT should be gauged after *subtracting* out 1 s from the data. Observe the close agreement between the two test formulations when the actual bias is identical to the design bias. Also, note that detection times increase as p decreases. Finally, the sharp increase in detection times across the 0.1719 bias level should be studied. A decrease in p or a slight increase in the design bias level could assure that bias levels below 0.1719 are not detected. Notice that for $p = 1\times10^{-30}$ and an actual bias of 0.15, a failure is declared only one time in 50 runs with each run being of 30-min duration.

Another interesting comparison is that between the Shiryayev SPRT and the direct comparison trigger needed to initiate the Wald SPRT. Reference 6 discusses the construction of the trigger based on desired false alarm and missed alarm probabilities. For equal alarm probabilities of 1×10^{-4}, a moving measurement interval six samples wide is employed to test for a failure by an average of the interval residuals. For an actual bias of 0.3438 deg/s, the trigger alone

Table 3 Influence of arbitrary failure time on average Shiryayev SPRT detection times

Analytic redundancy type—Translational kinematics, instrument tested—Normal accelerometer

Design bias level $\triangleq 6.4$ ft/s, Residual interval $\triangleq 70T$, $T \triangleq 0.0625$, $\sigma_A = 0.984$ ft/s^2, $\sigma_q = 0.04$ deg/s, $\sigma_M = 0.01$ Mach, $\sigma_\alpha = 0.191$ deg, $\sigma_{\text{residual}} = 4.4$ ft/s, $\pi = 1 \times 10^{-20}$, $p = 1 \times 10^{-10}$, $F_T = 0.9999$

Detection times are 100 run averages.

Time of actual bias insertion (Actual bias = 6.4 ft/s)	Detection time, s
0	9.14
0.5	9.98
1.0	11.86
1.5	12.73
2.0	13.26
2.5	13.48
3.0	13.78
3.5	13.91
4.0	13.91
$4.357 = 70T$	13.91
5.0	15.05
6.0	16.84
7.0	17.59
8.0	18.33
8.75	18.33

requires an average of 0.25 s to "detect" a failure after the failure is injected. For a failure bias of 0.25 deg/s, 0.73 s is required. No failure is "detected" for actual biases of 0.19, 0.15, and 0.13 deg/s. Comparison of these times with those of Table 1 shows that the Shiryayev test has faster detection performance except when p is extremely small.

B. Dissimilar Instrument Failure Detection Using Analytic Redundancy

Translational kinematics analytic redundancy is employed to test for the presence of normal accelerometer failures. The aircraft state for this analysis is generated by a control configured aircraft which is controlled by the compensator designed in Ref. 11. The flight condition of the aircraft is Mach 9 and 20,000 ft MSL. This aircraft is maneuvered between two sets of direct lift commands every 10 s such that A^z alternates between 32.2 and -32.2 ft/s^2 while q switches from 1.978 to -1.978 deg/s. The angle of attack remains near zero. With the noisy aircraft state available, Eq. (34) and (35) can be used to form the failure residual. An approximate residual variance of 19.36 ft^2/s^2 is obtained from the variance analysis discussed in Sec. IV.

Table 2 presents some Shirayev SPRT detection time data for a relatively small accelerometer design bias of 6.4 ft/s^2. The failure is injected at the beginning of the test and the maximum time limit on each run is 30 min. Data are given for various values of the residual interval and for four actual bias levels. Again notice that the SSPRT tends to not detect failures below one-half of the design bias level. Observe that the minimum detection time occurs for a residual interval of 70 samples when the actual bias failure is identical to the design bias level. Table 3 indicates that this detection time of 9.14 s can increase by as much as 2 s if the actual failure time occurs when the SSPRT is in the middle of a residual interval. Since a hardover failure of the accelerometer would have serious repercussions on the performance of the flight control system, a minimum wait of 4.375 s (70 samples) is undesirable. Hence, two tests should be implemented in parallel where one has a relatively small design bias and the other has a relatively large design bias. In this manner hardover failures can be detected quickly while smaller failures are detected in reasonable lengths of time. Table 4 gives data for design biases of 32.2 and 64.4 ft/s^2. The minimizing residual intervals are 20 and 10 samples, respectively. Observe that average detection times for failures at the design level are reduced to 1.45 and 0.78 s, respectively.

Table 4 Large failure detection using analytic redundancy and the Shiryayev SPRT

Analytic redundancy type—Translational kinematics, instrument test—Normal accelerometer

$\sigma_A = 0.984$ ft/s^2, $\alpha_q = 0.04$ deg/s, $\sigma_M = 0.01$ Mach, $\sigma_\alpha = 0.191$ deg, $\pi = 1 \times 10^{-20}$, $p = 1 \times 10^{-10}$, $F_T = 0.9999$, $\sigma_{\text{residual}} = 4.4$ ft/s

Failure injected at $t = 0$ s, $T = 0.0625$

Detection times are 100 run averages.

Design bias	Residual interval, T	Actual bias	Detection time, s
32.2	15	32.2	1.87
		64.4	0.94
32.2	20	32.2	1.45
		64.4	1.25
32.2	25	32.2	1.56
		64.4	1.56
32.2	30	32.2	1.88
		64.4	1.88
64.4	5	64.4	1.25
64.4	10	64.4	0.78
64.4	15	64.4	0.94
64.4	20	64.4	1.25

VI. Conclusion

The Shiryayev SPRT is derived and implemented in the detection of failures in two similar instruments (pitch rate gyros), and in a single instrument (normal accelerometer) through translational kinematics analytic redundancy. The derivation of the SPRT requires that the measurements be conditionally independent and that the parameters p, π, and L_T be chosen. However, the density function associated with the measurement can be quite general. Since a failure is modeled as a change in the measurement distribution, the assumed failure mode will clearly influence the density functions. The failure type assumed herein is that of a known constant bias, while the unfailed state is assumed to have no bias present. This flexibility in the choice of the measurement density functions also allows the test to be applied to the absolute value of the measurements, such that only one test is needed to detect either a positive or negative bias. Also, the Shiryayev test does not require a triggering test and is in continuous operation. Hence, the test is simply implemented and shows remarkable sensitivity in the detection time to failure biases of about half of the design bias. The simplicity of the Shiryayev test implementation is somewhat compromised in the dissimilar instrument failure detection case, since a scheme to increase the signal-to-noise ratio results in delayed detection times if a small design bias is employed. A second SPRT implemented with a large design bias and operated in parallel is recommended to reduce the detection times for hardover failures.

Appendix A: Derivation of the Recursion Formula for F_k

The recursion formula (2) is derived on the basis of the assumption of independence of the z_k's given in Sec. II.A. By beginning the derivation at stage 1, the result at stage k is obtained by induction. By Bayes rule at stage 1,

$$F_1 \triangleq P(\theta \leq t_1 / Z_1) = \frac{P(z_1/\theta \leq t_1) P(\theta \leq t_1)}{P(z_1)} \tag{A1}$$

By definition, the conditional probability that $z_1 \in (\rho_1, \rho_1 + dz_1)$ is

$$P(z_1/\theta \le t_1) \triangleq f_1(z_1)\,dz_1 \tag{A2}$$

where dz_1 is an infinitesimal increment, and

$$\begin{aligned} P(\theta \le t_1) &= P(\theta \le t_0) + P(\theta = t_1) \\ &= P(\theta \le t_0) + P(\theta = t_1/\theta > t_0)P(\theta > t_0) \\ &\quad + P(\theta = t_1/\theta \le t_0)P(\theta \le t_0) \\ &= \pi + p(1-\pi) + (0)\pi = \pi + p(1-\pi) \end{aligned} \tag{A3}$$

Also note that

$$\begin{aligned} P(z_1) &= P(z_1/\theta \le t_1)P(\theta \le t_1) + P(z_1/\theta > t_1)P(\theta > t_1) \\ &= f_1(z_1)\,dz_1[\pi + p(1-\pi)] + f_0(z_1)\,dz_1[(1-\pi)(1-p)] \end{aligned} \tag{A4}$$

The conditional probability F_1 becomes

$$F_1 = \frac{[\pi + p(1-\pi)]f_1(z_1)}{[\pi + p(1-\pi)]f_1(z_1) + f_0(z_1)[(1-\pi)(1-p)]} \tag{A5}$$

Note that the differential dz_1 cancels in Eq. (A5).

Again by using Bayes rule at stage 2,

$$F_2 \triangleq P(\theta \le t_2/Z_2) = \frac{P(Z_2/\theta \le t_2)P(\theta \le t_2)}{P(Z_2)} \tag{A6}$$

Since the measurements are conditionally independent and $Z_2 = \{z_1, z_2\}$, then

$$F_2 = \frac{P(z_2/\theta \le t_2)P(z_1/\theta \le t_2)P(\theta \le t_2)}{P(z_1)P(z_2/z_1)} \tag{A7}$$

Noting that

$$P(z_2/\theta \le t_2) \triangleq f_1(z_2)\,dz_2 \tag{A8}$$

and

$$P(z_1/\theta \le t_2) = P(\theta \le t_2/z_1)P(z_1)/P(\theta \le t_2) \tag{A9}$$

then Eq. (A7) becomes

$$F_2 = f_1(z_2)\,dz_2 P(\theta \le t_2/z_1)/P(z_2) \tag{A10}$$

This can be put into a form similar to Eq. (A5) by observing that

$$\begin{aligned} P(\theta \le t_2/z_1) &= P(\theta \le t_1/z_1) + P(\theta = t_2/z_1) \\ &= F_1 + p(1-F_1) \end{aligned} \tag{A11}$$

and by analogy with Eq. (A4),

$$\begin{aligned} P(z_2) &= f_1(z_2)\,dz_2[F_1 + p(1-F_1)] \\ &\quad + f_0(z_2)\,dz_2[(1-F_1)(1-p)] \end{aligned} \tag{A12}$$

Therefore, Eq. (A10) becomes

$$F_2 = \frac{[F_1 + p(1-F_1)]f_1(z_2)}{[F_1 + p(1-F_1)]f_1(z_2) + [(1-p)(1-F_1)]f_0(z_2)} \tag{A13}$$

Note that the differential dz_2 cancels in Eq. (A13). By induction the recursion relation for propagating F_k is given in Eq. (2).

Appendix B: Properties of the Optimal Return Function $J_k^*(\cdot)$

The following lemma, which is proved for the Wald SPRT in Ref. 7, is extended here to the Shiryayev SPRT.

Lemma: The functions $A_k(\cdot)$ are concave with respect to the probability $F \in [0,1]$, and additionally satisfy conditions

$$A_k(1) = 0 \qquad \text{for } k = 1,2,\ldots,N-1 \tag{B1}$$

$$A_{k-1}(F) \le A_k(F) \qquad \text{for } k = 2,3,\ldots,N-1 \text{ and all } F \in [0,1] \tag{B2}$$

Proof: Equation (B1) is easily shown by observing that if $F_k = 1$ for any k, then $F_{k+1} = 1$ by the recursion relation. Hence, the final A_k is written as

$$A_{N-1}(1) = \underset{z_N}{E}\,[(1-1)Q/Z_{N-1}] = 0 \tag{B3}$$

which implies that the next to last A_k becomes

$$\begin{aligned} A_{N-2}(1) &= \underset{z_{N-1}}{E}\,[\min[(1-1)Q, C + A_{N-1}(1)]/Z_{N-2}] \\ &= \underset{z_{N-1}}{E}\,[\min[0, C+0]/Z_{N-2}] = 0 \end{aligned} \tag{B4}$$

By induction from Eqs. (B3) and (B4), $A_k(1) = 0$ for $k = 1,2,\ldots,N-1$.

Equation (B2) can be proven by first recalling that t_N is assumed to be the final stage. Therefore, for each time stage prior to this the minimum cost chosen must be the $C + A_k(F_k)$ term so that the test can continue. The last and next to last A_k terms become

$$A_{N-1}(F_{N-1}) = \underset{z_N}{E}\,[(1-F_N)Q/Z_{N-1}] \tag{B5}$$

and

$$\begin{aligned} A_{N-2}(F_{N-2})\; &\underset{z_{N-1}}{E}\,[\min[1-F_{N-1})Q, C \\ &+ A_{N-1}(F_{N-1})]/Z_{N-2}] = \underset{z_{N-1}}{E}\,[C + A_{N-1}(F_{N-1})/Z_{N-2}] \end{aligned} \tag{B6}$$

Hence,

$$A_{N-2}(F_{N-2}) \le A_{N-1}(F_{N-2}) \triangleq \underset{z_{N-1}}{E}\,[(1-F_{N-1})Q/Z_{N-2}] \tag{B7}$$

Also observe that

$$A_{N-3}(F_{N-3}) = \underset{z_{N-2}}{E}\,[C + A_{N-2}(F_{N-2})/Z_{N-3}] \tag{B8}$$

and because of Eq. (B7)

$$A_{N-3}(F_{N-3}) \le \underset{z_{N-2}}{E}\,[C + A_{N-1}(F_{N-2})/Z_{N-3}] \triangleq A_{N-2}(F_{N-3}) \tag{B9}$$

Therefore, by induction from Eqs. (B7) and (B9), $A_{k-1}(F) \le A_k(F)$ for $k = 2,3,\ldots,N-1$ for all $F \in [0,1]$.

Finally, $A_k(F)$ must be shown to be concave for $k = 1,2,\ldots,N-1$ and $F \in [0,1]$. By inspection $J_N^*(F_N)$ is concave. Observe that $J_{N-1}^*(F_{N-1})$ is concave if $A_{N-1}(F_{N-1})$ is concave. Hence, by induction, $J_k^*(F_k)$ is concave if $A_k(F_k)$ is concave.

Denote the elements of the infinitely countable measurement space z_k as z_k^i, where $k=1,2,\ldots,N$ and $i=1,2,\ldots,\infty$. Hence, for all $k=1,2,\ldots,N-1$ from Eqs. (2), (6), and (7),

$$A_k(F_k)=\sum_{i=1}^{\infty}\{[(1-p)(1-F_k)f_0(z_{k+1}^i) + [F_k+p(1-F_k)]f_1(z_{k+1}^i)][J_{k+1}^*(F_{k+1}(z_{k+1}^i))]\} \quad \text{(B10)}$$

$A_k(F_k)$ will be concave if each term in the summation is concave. Therefore, only the concavity of

$$h^i(F_k)=\{[(1-p)(1-F_k)]f_0(z_{k+1}^i) + [F_k+p(1-F_k)]f_1(z_{k+1}^i)][J_{k+1}^*(F_{k+1}(z_{k+1}^i))]\}$$

$$i=1,2,\ldots,\infty \quad k=0,1,\ldots,N-2 \quad \text{(B11)}$$

need be shown.

A function is concave if all points of the function between two points of the function lie above a straight line connecting those two points for any choice of the two points. Therefore, it must be *shown* that $h^i(F_k)$ satisfies

$$\lambda h^i(F_k^1)+[1-\lambda]h^i(F_k^2)\le h^i(\lambda F_k^1+[1-\lambda]F_k^2) \quad \text{(B12)}$$

for every $\lambda\in[0,1]$ and $F^1, F^2\in[0,1]$. Define

$$\xi_k^1=[(1-p)(1-F_k^1)]f_0(z_{k+1}^i) + [F_k^1+p(1-F_k^1)]f_1(z_{k+1}^i) \quad \text{(B13)}$$

$$\xi_k^2=[(1-p)(1-F_k^2)]f_0(z_{k+1}^i) + [F_k^2+p(1-F_k^2)]f_1(z_{k+1}^i) \quad \text{(B14)}$$

Then Eq. (B12), using Eqs. (B11), (B13), and (B14) and after some algebraic manipulations becomes

$$\lambda\xi_k^1J_{k+1}^*(F_{k+1}^1)+(1-\lambda)\xi_k^2J_{k+1}^*(F_{k+1}^2) \le [\lambda\xi_k^1+(1-\lambda)\xi_k^2]J_{k+1}^*\left[\frac{(\lambda G_k^1+(1-\lambda)G_k^2)f_1}{\lambda\xi_k^1+(1-\lambda)\xi_k^2}\right] \quad \text{(B15)}$$

where $G_k\triangleq F_k+(1-F_k)p$ and the z_{k+1}^i dependence is suppressed. Note that

$$\frac{\lambda\xi_k^1G_k^1f_1}{\xi_k^1}+\frac{(1-\lambda)\xi_k^2G_k^2f_1}{\xi_k^2}=\lambda\xi_1^1F_{k+1}^1+(1-\lambda)\xi_k^2F_{k+1}^2 \quad \text{(B16)}$$

Therefore, Eq. (B15), using Eq. (B16), becomes

$$\frac{\lambda\xi_k^1}{\lambda\xi_k^1+(1-\lambda)\xi_k^2}J_{k+1}^*(F_{k+1}^1)+\frac{(1-\lambda)\xi^2{}_k}{\lambda\xi_k^1+(1-\lambda)\xi_k^2}J_{k+1}^*(F_{k+1}^2) \le J_{k+1}^*\left[\frac{\lambda\xi_k^1F_{k+1}^1+(1-\lambda)\xi_k^2F_{k+1}^2}{\lambda\xi_k^1+(1-\lambda)\xi_k^2}\right] \quad \text{(B17)}$$

Recall that $J_N^*(F_N)$ is known to be concave and, hence, satisfies

$$\gamma J_N^*(F_N^1)+[1-\gamma]J_N^*(F_N^2)\le J_N^*(\gamma F_N^1+[1-\gamma]F_N^2) \quad \text{(B18)}$$

For the N-1 stage, choose

$$\gamma=\frac{\lambda\xi_{N-1}^1}{\lambda\xi_{N-1}^1+[1-\lambda]\xi_{N-1}^2} \quad \text{(B19)}$$

This value of γ implies from Eq. (B17) that $h^i(F_{N-1})$ is concave and, hence, $A_{N-1}(F_{N-1})$ is concave. If $A_{N-1}(F_{N-1})$ is concave, then $J_{N-1}^*(F_{N-1})$ is concave. By induction, $A_k(F_k)$ is concave for $k=1,2,\ldots,N-1$.

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References

[1] Shiryayev, A.N., *Optimal Stopping Rules,* Springer-Verlag, New York, 1977.

[2] Deyst, J.J., Charles Stark Draper Laboratory, Cambridge, Mass., private communication, 1978.

[3] Willsky, A.S., "A Survey of Design Methods for Failure Detection in Dynamic Systems," *Automatica,* Vol. 12, 1976, pp. 601-611.

[4] Wald, A., *Sequential Analysis,* Wiley, New York, 1947, Chap. 3.

[5] Deckert, J.C., Desai, M.N., Deyst, J.J., and Willsky, A.S., "A Dual-Redundant Sensor Failure Detection Algorithm for the F8 Aircraft," *IEEE Transactions on Automatic Control,* Vol. AC-22, No. 5, Oct. 1977.

[6] Deckert, J.C., Desai, M.N., Deyst, J.J., and Willsky, A.S., "Reliable Dual-Redundant Sensor Failure Detection and Identification for the NASA F-8 DFBW Aircraft," Charles Stark Draper Lab., Cambridge, Mass., Rept. R-1077, May 1977.

[7] Bertsekas, D.P., *Dynamic Programming and Stochastic Control,* Academic Press, New York, 1976.

[8] Johnson, N.L. and Kotz, S., *Continuous Univariate Distributions—1,* Houghton-Mifflin, New York, 1970.

[9] Chien, T.T., "An Adaptive Technique for a Redundant Sensor Navigation System," Charles Stark Draper Lab., Cambridge, Mass., Rept. T-560, Feb. 1972.

[10] Chien, T.T. and Adams, M.B., "A Sequential Failure Detection Technique and Its Application," *IEEE Transactions on Automatic Control,* Vol. AC-21, Oct. 1976, pp. 750-757.

[11] Speyer, J.L., White, J.E., and Hull, D.G., "Modern Control Synthesis Applied to the Longitudinal Decoupled Motion of an Aircraft," *Proceedings of the 1981 AIAA Conference on Guidance and Control,* Albuquerque, N.M., 1981, pp. 378-389.